Histoire de l'analyse diophantienne classique. D'Abū Kāmil à Fermat by Roshdi Rashed

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This book presents a history of Diophantine analysis beginning with the late ninth century algebraist Abū Kāmil and continuing with al-Karajī, al-Samaw'al, al-Khāzin, al-Sijzī, Abū al-Jūd, Fibonacci, Ibn al-Haytham, al-Yazdī, and al-Khawwām. In the second half of the book, Rashed shifts to the European Renaissance and Early modern authors: Bombelli, Gosselin, Stevin, Viète, Bachet, and finally Fermat.

Diophantine analysis, according to Rashed, does not originate with Diophantus. This is a consequence of Rashed's claim that algebra was invented by al-Khwārizmī as a science of equations in the early ninth century. Since algebra is necessary for Diophantine analysis, Diophantus could not have practiced either one. Thus, the first algebraist after al-Khwārizmī to exhibit a collection of indeterminate problems gets the credit as the inventor of indeterminate analysis. That person is the Egyptian mathematician Abū Kāmil, who worked later in the ninth century.

There are many problems with this account, beginning with the fact that medieval Arabic and early modern European mathematicians unanimously recognized Diophantus as an algebraist. In my review of his *Abū Kāmil. Algèbre et analyse diophantienne* [Oaks 2014], I outlined how Rashed denies indeterminate analysis to Diophantus by emphasizing superficial differences with Abū Kāmil, and by distorting the premodern arithmetic and algebra by rewriting everything with modern algebraic symbols. Then, by interpreting Abū Kāmil's text through these symbols, he invokes a grossly anachronistic interpretation of the solutions in terms of modern projective geometry.

Rashed repeats this story in the first 35 pages of the volume under review and the problems continue as he progresses beyond Abū Kāmil. In particular, he

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continues to interpret medieval indeterminate analysis in terms of algebraic geometry. It was I. G. Bashmakova who first suggested such a reading for Diophantus' solutions [1966] and Rashed applies the same interpretation to the algebraists after Abū Kāmil, starting with al-Karājī.

To keep this review short, I will focus on Rashed's treatment of the late 16th century French mathematician François Viète [174–204]. Rashed's errors here are both mathematical and historical. I restrict myself to two topics: Rashed's misunderstanding of the nature of indeterminate problems and his anachronistic reading of Viète's theorems on triangles, this time inspired by a different paper by Bashmakova.

One of Rashed's key claims about Viète's indeterminate analysis makes no mathematical sense and is not supported in the texts. He writes that Viète's analysis

> *admet des solutions irrationnelles pour les problèmes indéterminés*. [200]
>
> admits irrational solutions to indeterminate problems.

But if the restriction of solutions to rationals is removed, the problems become trivial! There ceases to be any classification of numbers into squares, cubes, and so forth since these terms apply to *all* (positive) numbers. A look into Viète's indeterminate problems shows indeed that all solutions are rational.

Rashed's evidence for his claim comes from a passage that he cites from a scholium to Viète's first zetetic by the translator de Vaulézard:

> *Il convient remarquer en ce lieu, que ce Zététique-comme aussi la plupart des suivants, se peuvent non seulement appliquer à deux grandeurs ayant longueur seulement, comme sont les côtés: Mais généralement à toutes autres grandeurs*.... [176]
>
> It should be noted here that this Zetetic, like most of those that follow, can be applied not only to two magnitudes having length only, as are these sides, but generally to all other magnitudes....

Rashed summarizes:

> *Autrement dit, Viète étend le domaine de l'analyse indéterminée à d'autres corps de nombres que le corps des rationnels*.
>
> In other words, Viète extends the domain of indeterminate analysis to number fields other than the field of rationals.

Even with this snippet, Rashed should have seen that de Vaulézard was talking about extension to higher dimensional magnitudes and not to irrational numbers. In fact, de Vaulézard continues:

> *...pourveu que la somme et la diference proposée soient de mesme genre, soit que la question soit faite de plans, solides, plans plans, etc.* [de Vaulézard 1630]

> ...for seeing that the sum and the difference proposed are of the same kind, whether the question is about planes, solids, plano-planes, etc.

De Vaulézard's remark accompanies the first problem of book 1 of Viète's *Zeteticorum libri quinque*. All problems down to the middle of book 3 are determinate, including 'this problem' and 'most of those that follow'. Viète's numerical solutions to these determinate problems are often irrational and, of course, irrational solutions had been commonplace for such problems since at least the ninth century.

Rashed repeats his misinterpretation elsewhere in the chapter:

> *[Viète] donne une nouvelle orientation à l'analyse de Diophante (il n'exige pas, par exemple, que les solutions soient rationnelles).* [174]

and:

> *[Viète] a introduit les moyens et les techniques de l'algèbre dans l'étude des triangles rectangles, sans toutefois exiger que l'on obtienne des solutions rationnelles.* [204]

In his *Notae priores* (written in1593, published in 1631),Viète gives a series of propositions in which he relates the sides of a right triangle with acute angle θ with the sides of a right triangle with acute angle $n\theta$. Not surprisingly, his formulas on angular sections (*angulares sectiones*) are equivalent to trigonometric identities for $\cos(n\theta)$ and $\sin(n\theta)$. For example, given a single-angle triangle with base D, height B, and hypotenuse A, he expresses the base of the quadruple-angle triangle as $D^4-6B^2D^2+B^4$, its height as $4BD^3-4B^3D$, and its hypotenuse as A^4. This corresponds to our

$$\cos(4\theta)\& = \cos^4\theta - 6\sin^2\theta\cos^4\theta + \sin^4\theta \quad \text{and}$$
$$\sin(4\theta)\& = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta.$$

Despite the fact that Viète acknowledges only positive real numbers in his works, Rashed follows Bashmakova and Slavutin 1977 in recasting Viète's propositions in terms of operations on complex numbers. They note that Viète's formulas for the sides of the multiple-angle triangles can be read as the real and imaginary parts of $(x+iy)^n$, and even go so far as to suggest that that is what Viète really had in mind. But this is merely a coincidence, since the sine and cosine of multiple angles appear naturally in the polar formula:

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big).$$

Viète's propositions are in fact about triangles and triangles only.

So how does Rashed justify this interpretation if in Viète's time no one had yet worked out such calculations on complex numbers? He writes:

> *Cet inconvénient historique est compensé par l'avantage épistémique de conjuger les deux interprétations, algébrique et trigonométrique.* [204]
>
> This historical disadvantage is compensated by the epistemic advantage of combining the two interpretations, algebraic and trigonometric.

Two pages back he expressed the algebraic interpretation as a search for rational or irrational solutions to algebraic equations, formed mostly when studying triangles, and the trigonometric interpretation as 'an underlying search...for the trigonometric formulas':

> *À l'évidence, ce calcul admet deux lectures à la fois: recherche de solutions rationnelles ou irrationnelles des équations algébriques, formées pour la plupart lors de l'étude des triangles; et recherche sous-jacente, semble-t-il, des formules trigonométriques.* [202]

But what need would there be to combine these two interpretations? They are perfectly compatible as they stand, so there is no reason to impose a reading with complex numbers!

He continues his defense of this interpretation:

> *Certains historiens que rebute le recours à un autre langage—et à une autre mathématique—que celui de l'auteur ne manqueront pas de taxer cette interprétation d'anachronisme. Mais, si on la prend pour ce qu'elle est, c'est-à-dire l'instrument permettant de dévoiler le sens du phénomène étudié, que Viète ne percevait pas encore mais dont il pouvait avoir une certaine intuition, alors elle est la bienvenue. Mais ceci suppose que l'on ne prend pas l'instrument pour l'objet auquel il s'applique.* [204]
>
> Some historians who are put off by the use of another language—and another mathematics—than that of the author, are sure to charge this interpretation as anachronistic. But, if taken for what it is, which is to say, as the instrument allowing for the development of a sense of the phenomenon studied, that Viète does not perceive but of which he might have had some intuition, then it is

> welcome. But this assumes that one does not take the instrument for the object to which it is applied.

There is no evidence to back up Rashed's suggestion that Viète 'might have had some intuition' into raising $x + iy$ to powers and that he preferred to mask his discovery by presenting it with triangles instead!

There are other errors in the section on Viète that I cannot expose adequately in a short review, such as Rashed's attempt to link Viète's algebra with that of al-Khwārizmī, distancing both from Diophantus, or his misunderstanding of Diophantus' *eide* (species). And beyond Viète lies close to 100 pages on Fermat, which exhibit the same kinds of problems of interpretation.

The book could have been a handy introduction to early Diophantine analysis. But Rashed's misrepresentation of the history, together with his insistence on reading the premodern texts in terms of 20th-century mathematics, renders the whole project too misleading to be of any real use. This is too bad: the works of several Arabic authors in particular could have benefitted from a balanced treatment.

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