Angles et grandeur. D'Euclide à Kamāl al-Dīn al-Fārisī edited by Roshdi Rashed

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Angles et grandeur by Roshdi Rashed involves history, mathematics, and philosophy. The subject is angle as a magnitude and it is based on Arab manuscripts dating from the early period of Arab science (ninth century) until the period of the last great Arab scholars (14th century). Having a 706-page book dealing exclusively with angles may seem odd to the general reader and one may wonder what kind of interesting information may be contained in such a book. But skimming these pages, and especially the fascinating comments made by Rashed on the manuscripts which are published here for the first time, will show that the questions discussed are among the most fundamental of those concerning classical Greek mathematics and its continuation by Arab mathematicians. It will become clear after a thorough reading of this book that the questions about the notion of angle and magnitude that are addressed here lie at the heart of mathematics. These questions had tremendous repercussions in the late philosophical-mathematical literature and a real impact in the development of geometry.

The book starts with a general review of the questions raised by the notion of angle expressed in Euclid's *Elements*. This notion was considered from both mathematical and philosophical points of view in the writings of Plato and Aristotle, since mathematics and philosophy were intricately linked at that time. The fundamental idea of science, in particular, of mathematics, that arose before Plato and included the thorough investigation of the meaning of the words 'definition', 'axiom', 'common notion', and so forth, involves in an essential way the multifaceted discussion of the notion of angle. Whether angles, lines, and so on belong to the Aristotelian categories of quantity, quality, relation, or position; whether these are magnitudes and, if yes, whether

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they are *homogeneous* magnitudes; whether we can compare angles within a certain class and, if yes, what are these classes and what are the tools used in such a comparison; whether we may apply to angles the known operations (addition, multiplication, and so on), the theory of proportions, and so forth—all these questions are discussed at length in several of Aristotle's treatises; and they remained essential in mathematical thought for 2,000 years. One must bear in mind that these philosophical issues and questions were raised *because* of the notion of angle in geometry.

It is in Euclid's *Elements* that the angle finds its central place among the foundational notions that are at the basis of any treatise on plane and solid geometry. Arab mathematicians between the ninth and the 14th centuries considered this topic from both the mathematical and the philosophical points of view. They transformed it, made it their own, and developed it in a substantial way. This is what Rashed's book is about.

Before going into the details of the content of this book, let us recall a few facts concerning angles from Euclid's *Elements*.

In the *Elements*, angles are introduced in book 1. Right angles are mentioned at the level of the postulates. Postulate 4 reads 'All right angles are equal'. Angles next appear at the level of the definitions.

> Definition 1.8
> A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
>
> Definition 1.9
> And when the lines containing the angle are straight, the angle is called rectilineal. [Heath 1956, 1.153]

We deduce from def. 1.9 that there is more than one species of angle. Rectilineal angles form a special class of angles in which the lines containing the angle are straight. The other classes include the curvilineal and the mixed. Among the latter are the so-called contact- or hornlike angles (a term used by Proclus), which Euclid considers in *Elem.* 3.16:

> The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle. [Heath 1956, 2.37]

This is the angle made by a circle and a tangent straight line. There is also the so-called semicircle-angle that is introduced in the proof of the same proposition.

> I further say that the angle of the semicircle contained by the straight line *BA* [which is the diameter] and the circumference *CHA* is greater than any acute rectilineal angle. [Heath 1956, 2.38]

Solid angles appear in def. 11.11 and figure in props. 11.20–26. They are used in the proof of the fact that there are only five solid regular polyhedra [see addendum to prop. 13.18]. The next three definitions concern right, obtuse, and acute angles.

> Definition 1.10
> When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
>
> Definition 1.11
> An obtuse angle is an angle greater than a right angle.
>
> Definition 1.12
> An acute angle is an angle less than a right angle. [Heath 1956, 153]

However, there was a precise mathematical notion of angle before Euclid that can be traced back to the Presocratic philosophers. The Pythagoreans, back in the sixth century BC, had certainly a precise notions of plane and solid angle; and they used them in their mathematical discoveries, in particular, in their work on regular polygons and the classification of regular polyhedra.

After angles, we must talk about magnitude. In Euclid's *Elements*, magnitudes satisfy a certain number of axioms. For instance:

> Axiom 1.1
> Things which are equal to the same thing are also equal to one another.
>
> Axiom 1.5
> The whole is greater than the part. [Heath 1956, 155]

Lines (or line-segments) are examples of magnitudes: they can be compared and the theory of proportions applies to them. But can we compare magnitudes such as a line and a curve that are not homogeneous? If yes, how? Comparison by length will not be the solution. Indeed, the reader will notice that these questions were addressed several centuries before the invention of infinitesimal calculus and that the general notion of the length of a curve was far from being completely developed. Aristotle, in *Physics* 8 and other treatises, already addresses the difficulties encountered in comparing an arc of a circle with a straight line. There are, again in Greek philosophy, several classes of magnitudes. But to what classes of magnitudes do the various notions of angle belong? For instance, Euclid's book 5 deals with the so-called Archimedean magnitudes. Do we enter the realm of non-Archimedean geometry in order to develop the theory of angle? The magnitudes that Euclid considers include lines, areas, and solids. Dealing with angles is thus problematic. In Euclid's *Elements*, only magnitudes of the same kind are compared, added, subtracted, or multiplied by an integer. For instance, a surface cannot be compared to a line. Likewise, the theory of proportions developed in the *Elements* applies only to magnitudes.

It should also be recalled that in the *Elements* there are no computations of values of magnitudes like lines, areas, or angles. In fact, there is no computation of any distance, radius, or angle except for statements like 'the sum of the three angles in a triangle is equal to two right angles' [*Elem.* 1.32] or 'two circles are to each other like the square of their radii' [*Elem.* 12.2]. Quotients, products, and so forth of magnitudes are only compared but never computed. Furthermore, the language is geometrical. For instance, Euclid talks about the 'square on the side' and not the 'square of the side'. This point of view contrasts with that of Archimedes, who had a strong inclination for numerical computations. It is well known that he computed approximate values for π and areas under a parabola, for example.

Dealing with angles is more complicated than dealing with lines or areas. One reason is that the value of an angle in Euclid's *Elements* lies between 0 and π. So adding two angles might be problematic, if the result is greater than π. In this sense, the notion of angle does not satisfy the so-called Archimedean axiom. This was pointed out by ancient authors. Another difference is that the operations on angles cannot be made if the angles do not belong to the same class. Hence, the importance of a careful classification of angles. It is relatively easy to compare rectilineal angles. However, Euclid also considers angles which are not rectilineal: for instance, the angle of contact between a circle and a tangent. This is a mixed angle: one side is the arc of a circle and another one is a straight line. At the end of the proof of *Elem.* 1.16, Euclid declares that the contact- or hornlike angle is smaller than any rectilineal acute angle. He also shows, in the same proposition, that the semicircle-angle is greater than any rectilineal acute angle. What is the exact meaning of such statements? These are some of the questions that puzzled the ancient mathematicians.

Let us recall that Leibniz introduced in his work a class of numbers (the so-called infinitesimals, which he also called differentials) that he postulated to be greater than zero but smaller than any positive number. He also posited rules to manipulate them by addition, multiplication, and so on. In the period between Euclid and Leibniz, Arab mathematicians treated infinitesimals in their own way. This is one subject highlighted in the book under review. Questions on the 'inclination' between two curves, on how one computes angles, and how one compares them are the direct way to infinitesimal mathematics. Topological notions are also involved: to define an angle as a region bounded by two curves, one needs to make precise the notion of the 'boundary' of a region. Other important notions that appear in the context of angles include continuity, convexity, infinite division.... Some of the questions related to these notions were raised in very precise terms by Aristotle in various treatises and they became fundamental objects of investigation in the Western world, starting from the Renaissance, and found important development during the 17th century in the works of Galileo Galilei, Wallis, Hobbes, and other scientists that culminated in the works of Leibniz and Newton. All this is well known. It is much less known that these questions were thoroughly studied by Arab mathematicians working in Syria, Iraq, Egypt, and Spain and that their mathematical and philosophical development attained an extremely high level of scholarship between the ninth and the 15th centuries.

Let us come back to the book under review. This is the first thorough essay devoted to the work of Arab scholars on this subject. It contains an analysis of the Greek writings translated into Arab on the one hand and the original contributions of Arab mathematicians on the other. Arab texts and texts by Greek mathematicians available only in Arab translation are here translated into French and analyzed. Some of the authors of the Arab texts presented were well known in the later Latin world; we find among them Ibn al-Haytham (the famous astronomer, physicist, and mathematician known in the Latin world as Alhazen), Ibn Sīnā (Avicenna, the well-known physician and philosopher), and Naṣīr al-Dīn al-Tūsī (whose work on the problem of parallel lines was known and quoted by Wallis among others). Many readers will encounter for the first time the names of al-Nayrīzī, al-Anṭākī, Ibn Hūd, al-Sijzī, al-Samaw'al, al-Fārisī, al-Qūshjī, al-Abharī, and al-Shīrāzī. All of them were important mathematicians.

Some of the works among those that are presented in this book deserve to be especially highlighted. We mention here the work of Abū 'Ali al-Ḥasan ibn al-Ḥasan Ibn al-Haytham (d. after 1040). His profound work on infinitesimal mathematics, in continuation of the works of Archimedes and Apollonius, represents an epistemic turning point in the theory of angle.[1] Ibn al-Haytham applied his theory of infinitesimals to the setting of angles, in particular, for the comparison between a contact-angle and a rectilineal angle. His study uses the fact that these two 'magnitudes' do not satisfy *Elem.* 10.1 (they are not Archimedean). Among the arguments, Ibn al-Haytham introduced are two sequences, an increasing one and a decreasing one, the second one bounding the first one from above. With these two sequences, Ibn al-Haytham was able to compare infinitesimals [101]. It is in trying to resolve difficulties that appear in the *Elements* that Ibn al-Haytham wrote his two famous treatises, the *Explanation of the Postulates of the Book of Euclid* and the *Book on the Solution of Doubts Relative to the Book of Euclid on the* Elements *and the Explanation of Its Notions*. In these works, Ibn al-Haytham created a new geometry where the notions of angle and of superposition are primitive elements. From his point of view, the notion of equality (similarity) of lines and areas are based on motion—a notion avoided by Euclid (and prohibited by Aristotle, who considered motion as pertaining to physics rather than mathematics). Ibn al-Haytham addressed the difficulties that are inherent in applying these ideas to the notion of angles (in particular, to solid angle). His work was continued by several Arab mathematicians, including Naṣīr al-Dīn al-Tūsī (1201–1274) in his commentary on Archimedes' *Sphere and Cylinder* in which he addresses the question of the comparability of lines and curves, and of curvilinear angles. Naṣīr al-Dīn used in particular a notion of 'rolling onto each other' in comparing the lengths of curves [469ff].

It may be worth saying a few words on the modern period. Hilbert, in his *Foundations of Geometry* [1898], introduced the notion of angle in the setting of his congruence axioms (Group IV). These are the axioms of motion. Klein, in his *Elementary Mathematics from an Advanced Standpoint* [1908–1909]

[1] See Rashed 1996–2000 or 2011–2017 for an English translation.

discusses angles at length in relation with motion, in the third part of his essay titled *Systematic Discussion of Geometry and Its Foundation*. In Birkhoff's axiomatization of geometry [1932], which is based on the real number system (and which is, therefore, minimalistic), angles belong to the list of four undefined notions, the other three being point, line, and distance.

Let us now turn to the content of *Angles et grandeur*. Chapter 1 contains critical editions with historical and mathematical commentaries of important manuscripts that concern the notion of angle in the Euclidean tradition. The first manuscript is a text of *Elem.* 3.15,[2] edited from 10 different Arab manuscripts. These Arab versions of Euclid are particularly important in the present context because the Arab mathematicians whose texts are edited in the book under review relied on them. The text of *Elem.* 3.15 is followed by an excerpt of a commentary on the first book of the *Elements*, edited from the so-called Qum manuscript, by the Neoplatonist philosopher and mathematician Simplicius (*ca* 490–*ca* 560), who is also a famous commentator on Aristotle. The text concerns angles, their species, whether they are magnitudes, and whether they are qualities, for example. The author quotes his predecessors Apollonius and Aghānīs.[3] This text is followed by a commentary by Ibn al-Hātim al-Nayrīzī (d. *ca* 922) on *Elem.* 3.15. Then follows an anonymous manuscript, referred to as the *Lahore manuscript* and titled 'Treatise on the Angle', in which Euclid, Apollonius, Simplicius, and Aghānīs are again mentioned. It contains a wealth of mathematical proofs and technical remarks on the divisibility of various species of angles. This memoir ends with the words:

> These are things concerning angle that leave one puzzled, given that some of its states necessarily imply that it is a magnitude and others that it is not.

Chapter 1 also contains two other texts, comments on Euclid's *Elements* by al-Anṭākī (d. 987) and by Ibn Hūd (d. 1085), the latter extracted from his encyclopedia *al-Istikmāl*. This chapter, with its texts and the commentaries, gives an impressive overview of the rich subject of angles.

[2] *Elem.* 1.15 in the Arab manuscripts corresponds to 1.16 in Heiberg's edition and in Heath's translation.

[3] The latter is referred to on page 52 in an edited excerpt: 'My friend Aghānīs…'.

Chapter 2 concerns more especially the research conducted on the notion of magnitude, in particular, magnitudes that do not satisfy *Elem.* def. 5.4[4] as well as those that do not satisfy the so-called Eudoxus-Archimedes axiom, which is extensively used in book 12 of the *Elements*. Rashed recalls that al-Sijzī (middle of the 10th century) considered the question of angles in his *Introduction to the Science of Geometry*, in a treatise called *All the Figures that Arise from the Circle*, and in an epistle on *Elem.* 11.23 which concerns solid angles. In fact, al-Sijzī worked out a classification of curves into measurable and non-measurable according to their form and to whether one can use them in the theory of proportions. He applied the same criteria to the study of plane and solid angles. Al-Sijzī also considered non-planar curves. (It should be recalled that in Euclid's *Elements*, all curves are planar; there are no spatial curves.) Thus, al-Sijzī introduced new sorts of angles that do not satisfy *Elem.* prop. 5.4. He used the notion of 'equality in power' and a process called 'continuous variation of the tangent'.

Another major author considered in chapter 2 is Ibn al-Haytham, whom we have already mentioned. His research on the angle is also part of his contribution on isoperimetry and isepiphany, in which he developed a geometry where situation is combined with measurement and where he included angles among the primitive elements of geometry. His work is both a continuation and an outcome of the work of the Arab mathematicians of the two centuries that preceded him. His investigations related to angle are included notably in his two books on the explanation and the correction of Euclid's *Elements*, in which he considers all the basic questions, such as the existence, classification, nature, and homogeneity of angles. Ibn al-Haytham also considered planar angles on convex surfaces. He discussed extensively the relation between equality and superposition, and he introduced kinematic notions in that theory. Chapter 2 contains a critical edition of Ibn al-Haytham's work on *Elem.* 15 from his *Book on the Solution of Doubts Relative to the Book of Euclid on the Elements and the Explanation of its Notions*. This chapter also contains a critical edition of a text by the algebraist al-Samaw'al ibn Yayā al-Maghribī (d. 1175) titled *Epistle on the Angle of Contact*, in which

4 According to this definition,

> magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another. [Heath 1956, 114]

Rashed gives an explanation of the non-homogeneity and the non-comparability of figures, based on the example of the angle of contact. Together with Kamāl al-Dīn al Fārisī (d. 1319), al-Samaw'al is one of the successors of Ibn al-Haytham who continued his research on angles of contact.

Chapter 3 has a more philosophical flavor. It contains a letter addressed by Ibn Sīnā (980–1037) to another physician and philosopher, Abū Sahl al-Masīḥī, who was an erudite familiar with Greek science and literature. In this epistle, Ibn Sīnā makes a systematic exposition of the notion of angle (planar and solid) with an examination of the opinions of several of his predecessors. The chapter also contains an excerpt on angles from Ibn Sīnā's famous treatise *Al-shifā'* (*Fragment of Book IV, Fifth Chapter*), in which he discusses the question of which Aristotelian category the concept of angle belongs to. He considers that angles belong to both categories of quality and quantity. Notions like quantity, quality, relation, magnitude, figure, limit, and others are considered in their philosophical aspect. Infinite divisibility of angles is also discussed. An adequate specialized metaphysical vocabulary is used that involves the distinction between 'in itself', 'by essence', and 'by accident', for what concerns the fact that angles satisfy the Euclidean definition of magnitude. Chapter 3 also contains a 'Treatise on the Angle' by Kamāl al-Dīn al-Fārisī (1266–1319), a philosopher who commented on the works of Ibn Sīnā and Ibn al-Haytham. This treatise constitutes a synthesis of the knowledge of these two philosophers on the question, using again a philosophical language. The same chapter contains fragments from Ibn al-Haytham's *Explanation of the Postulates of the Book of Euclid* which are quoted by al-Fārisı. The chapter closes with the memoir *On the Contact Angle* by 'Alā' al-Dīn al-Qūshjī (1403–1474) in which Rashed considers issues related to the continuity of angles.

Chapter 4 concerns solid angles. We recall that the study of solid angles started with the Pythagoreans who investigated regular polyhedra. The subject is also discussed in Plato's *Timaeus*. Euclid used the theory of solid angles in book 11 of the *Elements* in order to classify regular polyhedra. The theory that he developed was pursued by Arab commentators, who considered cases that were not considered by Euclid (e.g., concave angles). They also developed rules for the comparison of solid angles. For instance, Al Sijzī considered solid angles not bounded by planes. In the 11th century, Ibn al-Haytham developed the theory of solid angles in his research on isoperimetry and equal surface areas. He was motivated by the problem of approximating the volume of the sphere by volumes of convex polyhedra in his infinitesimal approach to the sphere. In his works, solid angles are subject to the usual operations that apply to Archimedean magnitudes and to the theory of proportions. He used the work of Archimedes on the sphere but also conical projections, and spherical geometry. One may recall here that in the Western world and after the Hellenistic period, research on solid angles started only after the 17th century in works of Descartes followed by Euler, de Gua, Legendre, and Cauchy, for example.

Chapter 5 of Rashed's book contains critical editions of Arab versions of Euclid's props. 11.20–23, 11.26, from the same manuscripts used for Euclid's Proposition presented in Chapter 1. These are now the propositions that deal with solid angles. Then comes a text by al-Sijzī, his *Epistle to Resolve the Doubt Relative to the Twenty-Third Proposition of the Eleventh Book of the* Elements *and to Another of His Constructions*. This epistle is followed by Ibn al-Haytham's commentary on *Elem.* 11.23 in his *Book on the Solution of Doubts Relative to the Book of Euclid on the Elements and the Explanation of Its Notions*. It is followed by commentary by the 13th-century philosopher, mathematician, and astronomer al-Abharī on props. 11.22–23, extracted from his *Commentary on Euclid's* Elements and a commentary on the same propositions by Naṣīr al-Dīn al-Ṭūsī from his redaction of the *Elements*. Finally, the chapter contains an excerpt of a commentary by an anonymous writer on *Elem.* 11.23 from the Escorial Manuscript.

Chapter 5 focuses on texts concerning the comparability of angles. The general question is how to compare magnitudes while taking into account the Aristotelean ban of motion. The author reviews questions related to equality, superposition, congruence, similarity, and so on in the works of Euclid, Apollonius, Proclus, and their Arab successors. This chapter contains critical editions of fragments of the redaction of Archimedes' book *On the Sphere and on the Cylinder* by Naṣīr al-Dīn al-Ṭūsī and a *Treatise on Rolling Motion and on the Relation between the Rectilineal and the Curve* by Al-Shīrāzī (second part of the 13th century).

Rashed also includes the Arab translation by Ḥunayn ibn Isāq of the definitions of *Elem.* bk. 11 and an Arab-French glossary of words.

In conclusion, *Angles et grandeur* is extremely rich in historical as well as mathematical information. One can admire the texts by the various authors quoted for the clarity and precision in their mathematical language. The texts presented, most of them published here for the first time together with the commentaries by Roshdi Rashed, constitute a major contribution to our mathematical, historical, and philosophical literature.

There is much still to be done. As the author remarks, there is no critical edition of the various Arab translations and commentaries of the *Elements*. Doing such a work will be an essential step in the reconstruction of the original work of Euclid. We also learn from Rashed that there exist treasures of Arab manuscripts to be studied, for which there is an urgent need of historians who are knowledgeable in mathematics and in Arabic.

Roshdi Rashed possesses a broad knowledge in mathematics and history, a deep insight in the foundations of mathematics and the interrelations between the different fields of science, and an unusual ability to transmit the important Arab mathematical texts and to comment on them. By his industrious work, he has transformed the landscape of the history of Greek and Arab mathematics. His writings render an incomparable service to science and history.

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