
The Mathematics of the Heavens and the Earth: The Early History of Trigonometry by Glen van Brummelen

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An etymological transformation perhaps unrivaled in the history of mathematics is that of the evolution of the lexical term for sine. Often recounted,¹ the linguistic passage of this term begins in India (*jyā/jīva*), is subject to the methodical magic of the translator's pen as it traverses the Islamic Near East (*jaib*) and ends up in the Latin west (*sinus*) as we recognize it today. This passage reveals a mathematical concept that is diachronic and richly multicultural. Indeed, as its etymology reveals, any adequate account of the field of trigonometry of which sine is just a part must too follow this trajectory. And for the first time, this has been achieved in a single work. *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry* by Glen van Brummelen follows the history of one of the most familiar areas of mathematics—trigonometry—and van Brummelen is acutely aware of the heritage of this discipline,

a subject that is so pervasive that almost all of us see some of it during our high school education, one whose story goes back well into ancient times. This subject crosses most major cultures and places; indeed, it is not easy to identify societies that contributed significantly to science without using it in some way. [xi]

¹ See, for example, [Plofker 2009](#), 257. Van Brummelen partially follows this passage [138].

Accordingly, van Brummelen's considerations begin in the ancient Near East, Egypt, and ancient Greece; they continue with its emergence and development in India and the Islamic Near East, and its transmission to the European Middle Ages and the Renaissance.

Trigonometry is a term well familiar to students and scholars alike. It features in both pure and applied mathematics and is as commonplace to beginners as it is to experts. Thus, van Brummelen's book is a welcome addition to this high profile topic. No other scholarly work provides historical coverage, mathematical analysis, and reflective commentary devoted to this subject. His analysis offers translations of key primary source texts and thorough but accessible accounts of their mathematical content.

Van Brummelen's style is warm and inviting. The book is well set out, diagrams are carefully rendered, and primary source extracts are integrated seamlessly into the main text. He endeavors to provide numerous footnotes for the express purpose of supplying readers who want to delve more deeply into the mathematical history than the text allows with appropriate resources. The bibliography has over 600 entries.

This book fills a conspicuous gap in the field. Even recently, Eli Maor, in his preface, stated that his book is 'neither a textbook of trigonometry—of which there are many—nor a comprehensive history of the subject, of which there is almost none' [1998, xi]. With the publication of van Brummelen's latest contribution, scholars now have access to the first half of such a comprehensive history. Until now, texts like Maor's and strong entries on the history of trigonometry in popular mathematics history² have been the only sources to provide researchers with significant scholarship that traces the roots and development of trigonometry. As for his inspiration, van Brummelen acknowledges a clear debt to Anton Von Braunmühl [1900–1903], but it is clear that this work surpasses this classic in many ways.

From the outset, indeed, even in his title, van Brummelen conveys that this science was inspired by celestial musings as well as measuring and reckoning in the terrestrial realm. Because of these practical orientations, he includes a short introductory chapter with details on the essential and basic concepts of spherical astronomy.

² For example, survey texts such as [Katz 2008](#).

Here, he defines a variety of terms (set-off nicely in bold italics) and provides several diagrams depicting features needed for solar time-keeping, oblique angle of ascension, and calculating rising times.

Van Brummelen begins by carefully defining trigonometry. He provides two necessary conditions:

- (1) a standard quantitative measure of the inclination of one line to another; and
- (2) the capacity for, and interest in, calculating lengths of line segments.

In establishing these necessary conditions for considering trigonometry as a science, van Brummelen provides examples in which one or the other condition fails to exist. This gives justification for those sources that he has included and those that he has left out. For example, he describes why Plimpton 322 will not be included in the discussion—unlike Maor, for instance, who did take the position that Plimpton 322 was the first trigonometric table. Later in the work, he discusses this distinction when considering the *analemma* [172], which, according to the definition, does not satisfy the necessary conditions either though its importance to those involved in trigonometric activities is vital.³ He considers Egyptian Pyramid slope calculations and Babylonian astronomical computations as precursors, providing evidence of early interest in angle measurement. He focuses on the ancients' contributions to measuring length and angles and locates the first glimmers of trigonometry proper in the third century BC with Aristarchus and his consideration of the relative distances of the Sun, Moon, and Earth, and soon after with the work of Archimedes. Of particular interest in the chapter is the treatment of Archimedes' Theorem of the Broken Chord [31] and its influence on the work of al-Bīrūnī.

Next, van Brummelen discusses the contributions of Alexandrian Greece, with an emphasis on Hipparchus (e.g., his chord table), Ptolemy, Archimedes, Menelaus, and the emergence of spherical trigonometry. In turn, he considers how Greek and Babylonian astronomy influenced the development of Indian astronomy. In this

³ Van Brummelen states regarding the *analemma* that 'it seems fair not to consider it not as part of trigonometry as such, but rather as a mentoring older sibling.'

those that were ‘accurate’. Van Brummelen notes cases where practitioners ‘improved’ the rate of the convergence of series approximations by the addition of correctional terms. It has been argued that the new improved and ever increasingly accurate procedures were in fact not used by astronomers in their computations—their field could not take account of the level of precision that these iterative procedures offered them. It seems, then, that they were developed for their intrinsic mathematical interest, a supra-utilitarian motivation.

Following the contribution by Indian scholars, van Brummelen expertly describes the major achievements of Islamic Near Eastern mathematicians in the field of trigonometry—both planar and spherical. This is by far the most substantial chapter, and his presentation of sources otherwise not available and lucid mathematical analysis are impressive. The chapter begins with ibn Yunus’ improvement upon Ptolemy’s work to ‘build a better sine table’ [140]. It travels through the development of early spherical astronomy (e.g., an emphasis of graphical methods and *analemmas*) and the influence and use of Menelaus’ Theorem in Islam. As van Brummelen identifies them, the threads of transmission are a real tangle in the Islamic Near East and for a significant period of time the ‘*Almagest* was in the strange position of competing with the theories of its predecessor’ [137].

Mathematical highlights include ibn Yunus’ [141] and al-Kāshī’s computation of the sine of 1° [148], al-Ṭūsī’s work on spherical trigonometry [191], attempts to establish the direction of the *qibla* [195], and the application of trigonometry to astronomical instruments [209], to name a few. Van Brummelen establishes three main approaches to the study of spherical trigonometry in medieval Islam: the Greek approach, emblemized by Menelaus, the Indian approach with plane triangles on the sphere, and the tradition of the *analemma*. Van Brummelen argues here [167] and elsewhere that its use in India was overstated; however, it is certain that as more of the Islamic sources become better known, this relationship will become clearer.⁴ Van Brummelen is conscious of the extent of the Islamic empire and considers the regional variation of mathematical activity

⁴ For example, in an as yet unpublished manuscript of al-Khāzini, a 13th century astronomer who wrote the *zīj al-Sanjari*, refers to an analemma-like construction as ‘the Indian circle’.

chapter, van Brummelen describes the most significant Indian contributions to trigonometry and his coverage in this area is noteworthy in scope and detail. Here we are provided with an impressive range of authors and contrasting techniques which gives us a good sense of how trigonometry flourished at this time. Two important themes are well developed in the chapter: the development and improvement of sine tables and the establishment of trigonometric identities. Additionally, he considers the work by Indian astronomers to improve upon methods of spherical trigonometry, largely in the service of astronomy. Mathematical highlights include Nīlakaṇṭha's ability to accurately handle 'all ten cases of the astronomical triangle in one place' [129], the computation of the sine of 18° in the 12th century [105], Bhāskara II's ingenious relation of $\sin(A + B)$ [106–107], Brahmagupta's second-order interpolation scheme for approximating sines [111–112], as well as the Taylor series for trigonometric functions in Mādhava's Kerala school [113ff]. Van Brummelen makes the compelling comment 'Indian scientists wrote much more on their results than on their methods' [124]. This observation is indeed true; but the reasons for this circumstance are complex and fascinating, and deeper than he lets on. Calling it later Indian 'reticence' [105], van Brummelen remains silent on the broader ambient social and intellectual traditions that were responsible for this feature. Similarly, he deals with issues of intellectual transmission sensitively and soundly; but his statement that 'the main distinction between Greece and India is not in what they chose to study, but in what they chose to write' [95] again does not capture the richness of the Indian intellectual circumstances. Because of the oral tradition and other predominant aspects of society and culture, the transmission and subsequent reception of foreign ideas into India is nuanced and multidimensional.

Some further reflections about applications may have further enriched his coverage. Van Brummelen observes that these authors never left the astronomical content for more general mathematical discourse [132]. In fact, as Plofker [2009, 210ff] notes, trigonometry *per se* was a special application in astronomy of geometry and, as the texts themselves reveal, was not considered part of more abstract mathematics at all. On the subject of application, Indian practitioners discriminated between those results that were 'practical' and

in al-Andalus (Muslim Spain), which is an area rising in prominence in studies of the exact sciences of this area [217ff].

The book closes with a thorough treatment of the transmission of this study from the Arab world to the West. Van Brummelen explores subsequent activity until 1550 with the work of Rheticus, a fitting concluding point, as Rheticus' work, the *Canon doctrinae triangulorum* (1551), is significant in two ways: it relates trigonometric functions directly to angles (and not to circular arcs) in keeping with contemporary practice and it tabulates all six of the trigonometric functions that we recognize today. Navigation gave impetus to further developments; and the long and lengthy computations required in this practical application, as well as in astronomy and geodesy, encouraged exploiting the various relations between trigonometric functions to ease the burden of computation.⁵

The book finishes with a glimpse of the future. While van Brummelen notes the decline of one branch of inquiry, spherical trigonometry, the growing importance of trigonometry is established in different ways: the solutions of differential equations representing harmonic oscillations, hyperbolic trigonometric functions, Fourier series, and infinite series, to name a few. Furthermore, trigonometry directed mathematicians' attention towards fundamental, more general notions in mathematics, such as continuity, functions, series, and limiting concepts. These topics are promised in a sequel to this volume.

Just as van Brummelen intended, this book will have wide appeal, for students, researchers, and teachers of history and/or trigonometry. The excerpts selected are balanced and their significances well articulated. As well as giving many vital details that have shaped this discipline, he has made some important observations about the transmission of mathematical ideas. It is a book written by an expert after many years of exposure to individual sources and in this way van Brummelen uniquely advances the field. This book will no doubt become a necessary addition to the libraries of mathematicians and historians alike. We look forward to the sequel with great anticipation.

⁵ Note, for example, the technique of *prosthaphaeresis* [264].

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