Arithmetic in Sixteenth-Century Muscovy by Mark A. Tsayger

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Arithmetic in Sixteenth-Century Muscovy by Mark Tsayger is dedicated to an important, complicated, and open research topic in the history of Russian mathematics—the *soshny* fractions.[2] I recall how, at one of the meetings of the Seminar on the History of Mathematics and Mechanics at Moscow State University, one well-known and respected scholar, an expert in this area and one of the presenters at the seminar, Professor Adolf Yushkevich (1906–1993), observed that the system of fractions obtained by successive divisions by two of fourths (*chets*) and thirds (*trets*) merits its own special research. Thus, it is without a doubt that Tsayger's recent work into this area is of substantial interest.

Yushkevich, in his fundamental book on the history of mathematics in Russia [1968, 16], wrote:

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[2] MW: the adjective 'soshny' in Russian refers to the tax unit, the *sokha*, corresponding to a variable amount of tilled land in 16th–century Moscow. In the existing literature, 'sokha' has been translated as 'plough'.

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> These two kinds of fractions played a role in the collection of taxes and constituted an important part of the *soshnoe pis'mo* (tillage accounting), which was the system describing the totality of methods for collecting taxes on parcels of land in the 16th and 17th centuries.

Tsayger's research on this topic is undoubtedly a significant milestone. One of the book's positive characteristics is the author's attempt to relate its subject matter partially to Old Russian mathematical culture. To this end, the book starts with a description of the archaic 'alphabetical' numerals, which remained in use in Rus' until the 18th century, when they were replaced with the modern Hindu-Arabic numeral system under one of Peter the Great's reforms. (As it is generally known, the latter form of numeration had already entered Russian life by the 17th century, chiefly through the handwritten version of *Numeral Calculating Wisdom* (*Tsifirnoi schetnoi mudrosti*), a manual on calculation arithmetic.)

Tsayger also discusses the original Russian method of denoting numbers 'in grids' ('v reshyotkakh'). This method was first elucidated in the historiography by the famous Russian scholar and well-known church figure, Metropolitan Evgeny Bolkhovitinov, in the first Russian-authored work on the history of mathematics [1813]. The method of denoting numbers 'in grids' has not been sufficiently researched. Indeed, discussion of this method of representing numbers is absent in the aforementioned book of Yushkevich, in the well-known mathematical history by B. V. Gnedenko [1946], and in the four-part *The History of Russian Mathematics* [Shtokalo 1966–1970]. This absence increases the value of Tsayger's book, in which the grid notation is considered in the requisite detail necessary for analysis of the sources regarding the *soshnoe pis'mo*.

The main sources of Tsayger's research go beyond the 'Books of Tillage Accounting' ('Knigi soshonogo pis'mo'), used during the 17th and 18th centuries, and also include the aforementioned *Numeral Calculating Wisdom* and *Arithmetic* (*Arithmetica*). These manuals have survived as handwritten texts and each copy is unique. In a number of them, there is a special section devoted to the so-called *doschany schot* (board abacus), a precursor to the well-known Russian/clerical

abacus or *schoty*.[3] The principles of constructing a *doschany schot* were described and researched by the famous historian-numismatist Ivan Spassky [1952].

The first *doschany schot* was a rather cumbersome instrument comprised of four counting fields. These counting fields were separated by wooden frames, to which were fastened rods or cords lined with counting beads for the calculation of monetary and fiscal sums and measurements involving volumes and weights. As opposed to the modern *schoty*, the *doschany schot* had lower counting fields made up of rods or cords for calculations involving quarters and thirds along with their binary divisions.[4] In spite of the attention many historians of science have given to the *doschany schot*, it still has not been fully studied.

To a certain degree, Boris Gnedenko's comment back in 1946 is reasonably fair: 'By all appearances, the explanations of the uses of the *doschaty schot*[5] have been lost; only rather unclear descriptions of its implements survive' [1946, 48]. Thus, Tsayger's attempt to penetrate the secrets of the *doschany schot* merits the attention of historical-mathematical science. It is symptomatic that the author himself modestly believes that he has not succeeded in illuminating all of these secrets: he writes,

> Some special letter combinations in these schemes have up until now not lent themselves to deciphering.... Many of the topics which we have discussed are more assumption than demonstrated facts. [68]

Nonetheless, Tsayger's method of analysis is quite scientific and well deserving of further development and use by other researchers.

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[3] MW: the Russian abacus is known both in the Russian and English literature as a *schoty*. It will be referred to as a *schot* when referencing the historical instrument and as a *schoty* when referencing the modern abacus used widely in Russia. In anticipation of the reoccurrence of these terms, it may be useful to provide their definitions here: 'doschany schot' designates a board abacus; 'dschitsi schotnie', auxiliary calculating tablets; and 'schot kost'mi', the abacus of beads (or loose abacus).

[4] MW: that is, in the case of thirds: sixths, twelfths, twenty-fourths, and so on.

[5] This term is used by Gnedenko for the method of calculation that we are currently examining, 'doschaty schot' (instead of 'doschany schot').

The reason for this lies in the fact that Tsayger 'takes a stab'[6] at explaining the arithmetical 'technology' in Russian governmental records and *fisc* or treasury holdings in the 16th century; there are essentially no direct resources about this calculating technology. For this reason, he uses a method of reconstruction that accepts as a basis the so-called *dschitsi schotnie* (auxiliary calculating tablets), which are found in Russian mathematical manuscripts from the 17th century (in even earlier sources they are completely absent). Historians of mathematics have analyzed the *dschitsi schotnie* before Tsayger. Indeed, the four-volume *The History of Russian Mathematics* interprets the *dschitsi schotnie* in the following manner:

> In some of the manuscripts from the 17th century, sketches are found with depictions of the *dschitsi schotnie*, which appear to be variations of the *doschany schot* of the 17th century.

Consequently, in that publication, *dschitsi schotnie* and the *doschany schot* are treated as interconnected but different mathematical phenomena. This relationship is also confirmed in Tsayger's book with his description of the distinction between the *dschitsi schotnie* and the *doschany schot*:

> *Dschitsi schotnie* differ from the *doschany schot* only in that they consist of counting fields divided into 13 or 12 straight lines, from which the six lower ones are divided in half. [Shtokalo 1966–1970, 115]

Spassky considered the *dschitsi schotnie* to be a prototype of the *doschany schot*. This follows from the caption that he placed with the depiction of the *dschitsi schotnie* taken from a Russian mathematical manuscript of the 17th century:

> Draft of the *doschany schot* in the handwritten *Arithmetic* from the middle of the 17th century. [1970, 124]

Spassky suggested that in the 16th and 17th centuries the *doschany schot* coexisted with the more archaic Russian *schot*—the *schot kost'mi* (abacus of beads or loose abacus), which did not have frames and whose beads were not threaded but loose. He believed that the

[6] MW: I have retained here, and throughout, Simonov's quotes that appear in the original review. In this instance, Simonov uses the verb 'zamaxnut'sya', which evokes his sense of the bold character of Tsayger's interpretive work.

final step in the transition to the *doschany schot* in Russia took place before the middle of the 17th century:

> But sometime before the middle of the 17th century, the *doschany schot* prevailed and became universal and widespread throughout the entire territory of the Russian state. [1970, 123]

Tsayger characterizes the *dschitsi schotnie* in the following way:

> Now we can answer the question 'What is a *dschitsa schotnaya*?'. It is a scheme painted on an auxiliary tablet depicting the way in which to mark a table for a *schot kost'mi*.... In the 16th century, these auxiliary tablets were indispensable for the persons performing calculations, preventing them from making mistakes when transferring the result of the calculation to paper in Slavonic numerals. Evidently, even later in the 17th century when many chalk-lined accounting tables were replaced by the *doschany schot*, and Slavonic numerals were replaced by Arabic numerals, the need for *dschitsi schotnie* did not immediately decline. [42]

Therefore, Tsayger, unlike Spassky, does not consider the *dschitsi schotnie* to be drafts of the *doschany schot* but rather auxiliary calculating references used with a *schot kost'mi*, which itself preceded calculating instruments (*schoty*) taking the name 'doschany schot'.

Tsayger's opinion has something in common with the point of view expressed in the multi-volume *The History of Russian Mathematics* [Shtokalo 1966–1970] that the *dschitsi schotnie* and *doschany schot* are alike but represent different methods of calculation. Moreover, the 1966 edition does not identify concretely to which method of calculation the *dschitsi schotnie* correspond. By the way, here there is no mention of the *schot kost'mi*: only the Western European 'line abacus' ('schot na liniyax') is considered, which is associated with the *schot kost'mi ili penyazi* (counter of beads or money) of the Russian mathematical manuscripts of the 17th and 18th centuries. (Actually, in the aforementioned title, there is a discussion of the 'line abacus'. However, Spassky believed that only the last part of the name ('schot...penyazi') corresponded to it, while the beginning part ('schot kost'mi') pertained to the original Russian *schot* in distinction from the line abacus.)

The history of the *doschany schot* is not clear. All the more interesting then, is Tsayger's attempt to investigate it. Foreigners who were living in or visiting Russia in the 16th and 17th centuries mentioned Russian calculators using the pits of fruits (plum and cherry) for counting rather than an instrument with wooden bars and threaded beads, that is, they were not using a *doschany schot*. To a certain extent, this contradicts Spassky's opinion that 'by the 17th century, the *doschany schot* (*schoty*) took over and became widespread throughout the entire territory of the Russian state.' If this were the case, the *doschany schot* should have had a certain degree of prevalence in Russia in the 16th century and the first half of the 17th century, and it is unclear how foreign observers did not notice it. Everything fits however, if we assume that foreigners observed a different kind of Russian *schot*—the *schot kost'mi*—to the extent that the *doschany schot* still was not in mass usage, and maybe it had such a limited distribution it was as if it had not existed at all.

Tsayger's idea that the *dschitsi schotnie* reflect the *schot kost'mi*, which itself preceded the *doschany schot*, may prove fruitful to mathematical history since there are really no other sources about the *schot kost'mi*. Almost every depiction of the *dschitsi schotnie*, of which there are many in Russian mathematical manuscripts, adds something to our understanding of the object and the individual calculating characteristics of the *schot kost'mi*.

Firstly, these manuscripts indicate that the *schot kost'mi* was constructed in a decimal system. We can conclude this because in the sketches of the *dschitsi schotnie* the counting pieces (beads) are depicted in quantities of 10s (rarely in nines) on each complete calculation level. Secondly, in this kind of *schot*, beads were used unthreaded or loose. Thus, they were drawn, as a rule, lying on the lines of the *schot* and not threaded through them. True, occasionally but rarely, one finds depictions with threaded beads, which could say something about the influence of the *doschany schot*. Thirdly, in some *dschitsi schotnie*, archaic 'alphabetical' numerals are used and in others, modern (Hindu-Arabic) numerals. This shows that *schot kost'mi* may date back to the 16th century, when the modern system of writing numbers began to take the place of the 'alphabetical' numeration. Fourthly, the lower portion of the *dschitsi schotnie* was divided into two parts for fourths and thirds, which were constructed

by binary principles. This speaks to the fact that the corresponding binary fractions had entered the *schot kost'mi*.

Not long ago, discoveries were produced (about which Tsayger could not have known before the time of this book review) showing that binary fractions with a basis of half fourths and half thirds were apparently already in use in Rus' in the 16th century. This is evidenced in the deciphering of an Old Russian text (from 16th-century records), in which the corresponding system of fractions by halves was used with fractional divisions of time: half quarter past 5, one and a half quarter past 8, one and a half quarter till 11, a half quarter past half till 11, half minus half a quarter till 2, and so on [Simonov 2009, 106–108]. Indirectly, this fact supports Tsayger's hypothesis that the *dschitsi schotnie* date back to the 16th century. They might even date back to the border between the 15th and 16th centuries, if we factor in the date of 1495, which is assigned to the convoy of some of the texts which accompanied the Russian calculations of fractions of the hours [Simonov 2009, 108].

The uncovering of the specifics of Old Russian counting in the 'soshny fractions' system is important and substantial material in Tsayger's book. Spassky [1970, 123] wrote the following about this problem:

> There were special conversion tables in the instructions for Russian 'accountants', which allowed them to bring fractions of either base [RS: fourths and thirds] to a common denominator. It is remarkable how this monetary counting served them: it appears that one can express any fraction of either kind in the form of a monetary sum, after which adding or subtracting thirds and fourths is as easy as can be.

Those auxiliary resources, which Spassky calls 'special conversion tables', can produce case-specific formulas. Tsayger reproduces one of these formulas, which in the language of Old Russian *soshny* fractions sounds like this:

> A *chet'* [fourth] and a half *chet'* and a half-half *tret'* and a half-half-half *tret'* [third], sums to a *tret'* and a half *tret'* of a *sokha*.

In modern fractional notation, it can be expressed in the following equality:

$$1/4 + 1/8 + 1/12 + 1/24 = 1/3 + 1/6.$$

The purpose of the equality is the conversion of one *soshny* fraction into another, which was needed for the rationalization of calculations associated with the collection of tax on holdings. These holdings consisted of privately owned plots of land of varying sizes and values (ploughed fields, woods, hayfields, and so forth.) With this, the problem of whether the calculations were correct was also solved, and it was for this reason that a fast mechanism existed to verify them.

For example, the so-called 'Moscow counting' ('Moskovsky schot'), which Tsayger describes in his book, used such a verification. At the foundation of this method lies the definition of the calculating unit, the 8 *altyn*,[7] equaling 48 *dengas*. The author renders its meaning as such:

> To the extent that 8 *altyns* contained 48 *dengas*, the significance of thirds and fourths would seem to increases 48 times. The result is that the fractional summands seem to transform into whole numbers, with which the service class (*sluzhilie lyudi*) knew how to operate. After receiving the final result, it turns back into a fraction by that very same principle (i.e., out of the identity that one equals 8 *altyns*). [48]

To check the reproduced equation (assertion) we need to replace the one in the numerator with 8 (*altyns*) or 48 (*dengas*), put it in terms of uniform monetary units, and check the arithmetic of the equation. In this case, following Tsayger's calculations, we get:

> In the left hand side of the statement:
> quarter = 2 *altyns*
> half quarter = 1 *altyn*
> half-half third = 4 *dengas*
> half-half-half third = 2 *dengas*.
>
> $$2\,altyns + 1\,altyn + 4\,dengas + 2\,dengas =$$
> $$3\,altyns + 6\,dengas = 4\,altyns.$$

[7] MW: the *altyn* and *denga* (plural in Russian *dengi*) were Russian monetary denominations used widely before, and in the case of the *altyn* into, the Soviet period. 'Dengi' is now the Russian word for money, while the meaning of the word 'altyn' has been the subject of speculation by academics, some suggesting that it comes from the Tatar word for 6.

> In the right hand side of the statement:
> third = 2 *altyns* 4 *degas*
> half third = 1 *altyn* 2 *dengas*.
>
> $$2\,altyns + 4\,dengas + 1\,altyn\,2\,dengas =$$
> $$3\,altyns + 6\,dengas = 4\,altyns.$$
>
> The left hand side equals the right, i.e., the statement is correct. [52]

Actually it was a very simple method of verification and not borrowed at that but rather developed from an original Old Russian financial basis.

It seems that the designation of the method as 'Moscow' allows it to be dated. Divide 48 by 8, and we receive 6; this means that 1 *altyn* equaled 6 *dengas*. From the *Dictionary of Numismatics*, one can find that the *altyn* most likely had the above meaning only after the 14th century:

> *Altyn*, a Russian countable-monetary unit of the 14th century, equaling 6 *dengas*, later 3 kopecks. [Fengler, Gierow, and Unger 1993, 12]

The history of money circulation in Rus' shows that this data is not always absolutely exact. In actuality, in the last decade of the 14th to the first half of the 15th century, the *altyn* in Moscow equaled 3 *dengas*. Moreover, the minting of coins in Moscow and other Russian principalities was not unified. Such an unfavorable state for the development of the Russian economy ended partially, when, in 1420 in Novgorod, the Moscow norm of minting was accepted. After that, accounting was divided between the Novgorod and Moscow styles, a division that took place during the final years of the reign of the Grand Prince of Moscow, Vasily the Blind (1415–1462). The Novgorod *denga* retained its weight, adopted in 1420, while the Moscow *denga* became equal to half a Novgorod *denga* [Yanin 1970]. It follows, that only from this time the relationship 1 *altyn* = 6 *dengas* appeared, which had its primary use for the lighter Moscow *dengas*. It was named 'moskovka':

> 'Moskovka', beginning with the 16th century—name for the Moscow *denga*, which, although minted to a modest degree

> during the 16th and 17th centuries, was mentioned in a majority of commerce-related acts from that period. [Fengler, Gierow, and Unger 1993, 208].

It is possible that the name of the accounting as 'Moscow' follows from the connection with the *moskovka* currency, in which case the appearance and distribution of the 'Moscow counting' would date back to the 16th and 17th centuries.

There is some basis for more precisely defining the wide period of the first appearance of the 'Moscow counting'. The importance lies in that the *novgorodka*, a type of *denga* minted in Novgorod after 1534, changed its name and place of distribution: it became known as the kopeck and started being minted and used in Moscow:

> Kopeck, Russian silver coin, minted starting in 1534; its weight equaled the weight of a Novgorod *denga*, or *novgorodka*, which came into use in Moscow after the conquering of Novgorod by Ivan III (1462–1505) in 1478. [Fengler, Gierow, and Unger 1993, 141: cf. Spassky 1970, 111–113]

Consequently, the period between the last years of the reign of Vasily II, approximately 1462, and 1534, the year of the establishment of one united monetary system for the Russian state, was a much more convenient and suitable time for the appearance of the 'Moscow counting' (the name of which was formed from 'moskovka'). This was because, after 1534, the economical and political reasons for the division between Moscow and Novgorod minting systems had already disappeared, although the names of obsolescent monetary synonyms could have remained in usage for a long time.

Thus, there is some foundation to consider that the 'Moscow counting' could have appeared in the last decades of the 15th and early decades of the 16th centuries. I went into such extensive detail about the dating of the 'Moscow counting' because it is impossible to rule out the origins of the *schot kost'mi* (in the version which Tsayger reconstructs for the 16th century) from the 15th century. This has an important meaning for Tsayger's basic argument for the legitimacy of his reconstruction of the arithmetic knowledge of Muscovy in the 16th century based on sources (*dschitsi schotnie*, etc.) preserved in the records of a later period, the 17th and 18th centuries.

Of the many different arithmetic methods of 16th-century Russia reconstructed by Tsayger with which one may become directly

acquainted by his book, I would like to touch upon the question related to approximations. The author, having dug deeper into the calculating material, noticed something remarkable in it: Old Russian mathematicians dealing with values less than $\frac{1}{48}$, 'simply threw them out, assuming that taking them into account would have no effect on the result' [57]. That being said, it is classified as a defect of the method ('the arithmetic had one disadvantage' [57]). Here our respected author acts as an expert who evaluates the phenomenon by its mathematical merits and not by the historical context of its emergence and functioning. Taking into account that the discussion is about a special arithmetic—the *soshny* fractions—it follows that attention should be given to the historical economic side of that arithmetic. The *soshnoe pis'mo* was intended for the realization of the governmental fiscal project, the Bol'shaya Sokha (the Great Plough) carried out and developed practically 'from scratch' by Ivan the Terrible in the middle of the 16th century regarding the levying of tax on huge land holdings of Russia. The task of collecting everything up to the last kopeck would require enormous expenses of resources and time for the education (which would have had to include mathematical training) of a huge army of tax collectors and for the training and maintenance of a security force for their protection and the wresting out of debts. With such a perspective, the activity of the financial service would get so bogged down in problems and stretch out into such long years that it would never reach its desired purpose.

Under these conditions it was better to view the task of realizing the Bol'shaya Sokha project as an optimization problem: 'How to reach the maximum potential tax revenue with the minimum costs?' *A priori*, for example, the problem could be solved by the development of simple methods for the approximate calculations of tax levies, whereby weakening the accuracy of the estimate would save one time and expenditure of mental energy. Alongside this should have been the ability to verify the calculations quickly and simply so as to reduce the number of taxpayer appeals and complaints about the dishonesty of the accountants and clerks. Perhaps Tsayger's reconstruction of

the specifics of Russian *soshny* arithmetic from the 16th century is valuable, most of all, in that it answers this important question.

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